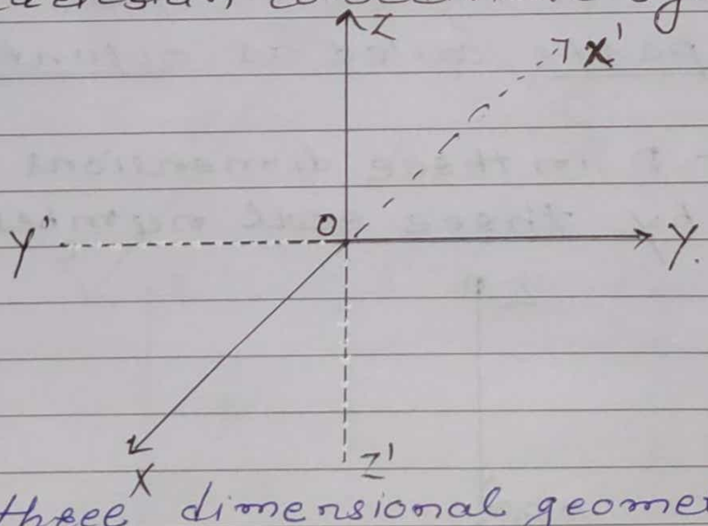
Three Dimensional Co-ordinate Geometry

* We know that to describe a point in two dimensions we use ordered pair (x, y) or (r, θ). where (x, y) are called cartesian co-ordinates of a point & (r, θ) are called polar co-ordinates.
  In this chapter we study the description of a point in space. There are three simple methods of describing a point in space.
1) The cartesian co-ordinate system (x, y, z)
2) Spherical polar co-ordinate system (r, θ, ϕ).
3) Cylindrical polar co-ordinate system (ρ, ϕ, θ).

*1 The cartesian co-ordinate system :



In three dimensional geometry we have three mutually perpendicular lines $X'OX$, $Y'OY$, $Z'OZ$ intersecting at O. These lines are known as co-ordinate axes

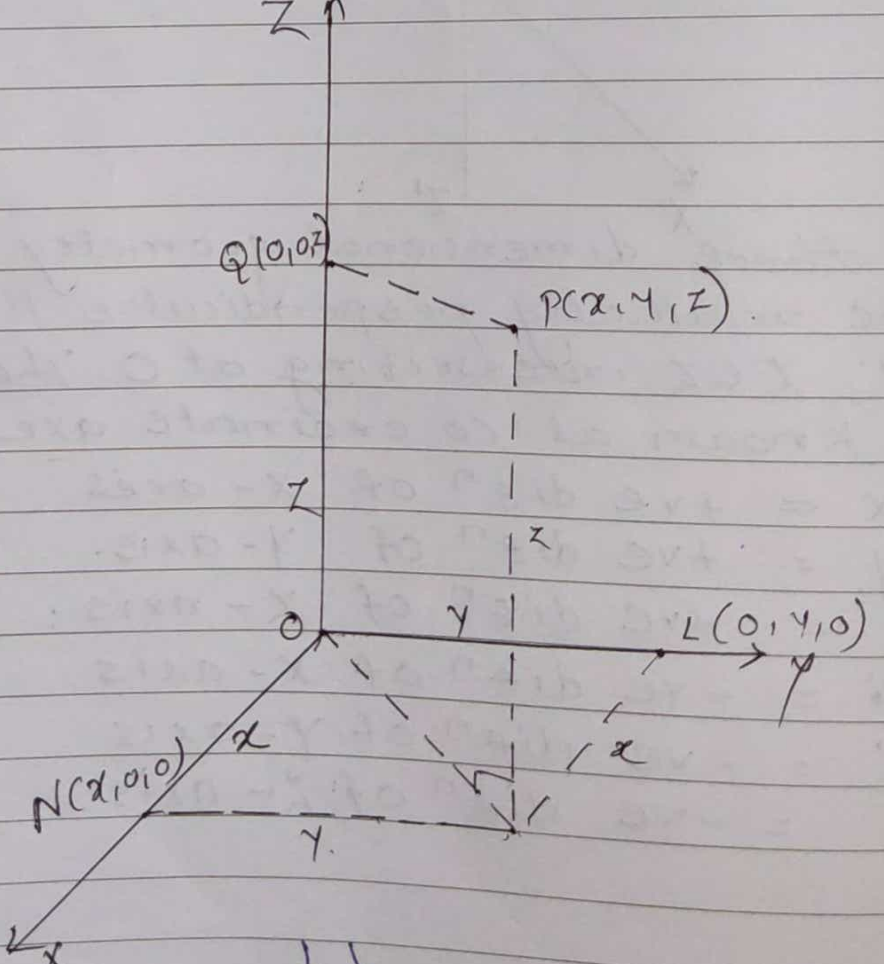
OX = +ve dir^n of X-axes
OY = +ve dir^n of Y-axis.
OZ = +ve dir^n of Z-axis.
& OX' = -ve dir^n of X-axis
OY' = -ve dir^n of Y-axis
OZ' = -ve dir^n of Z-axis.

- The plane in which X-axis, Y-axis lies is XY plane.
- The plane in which Y-axis, Z-axis lies is YZ plane.
- The plane in which Z-axis, X-axis lies is ZX plane.

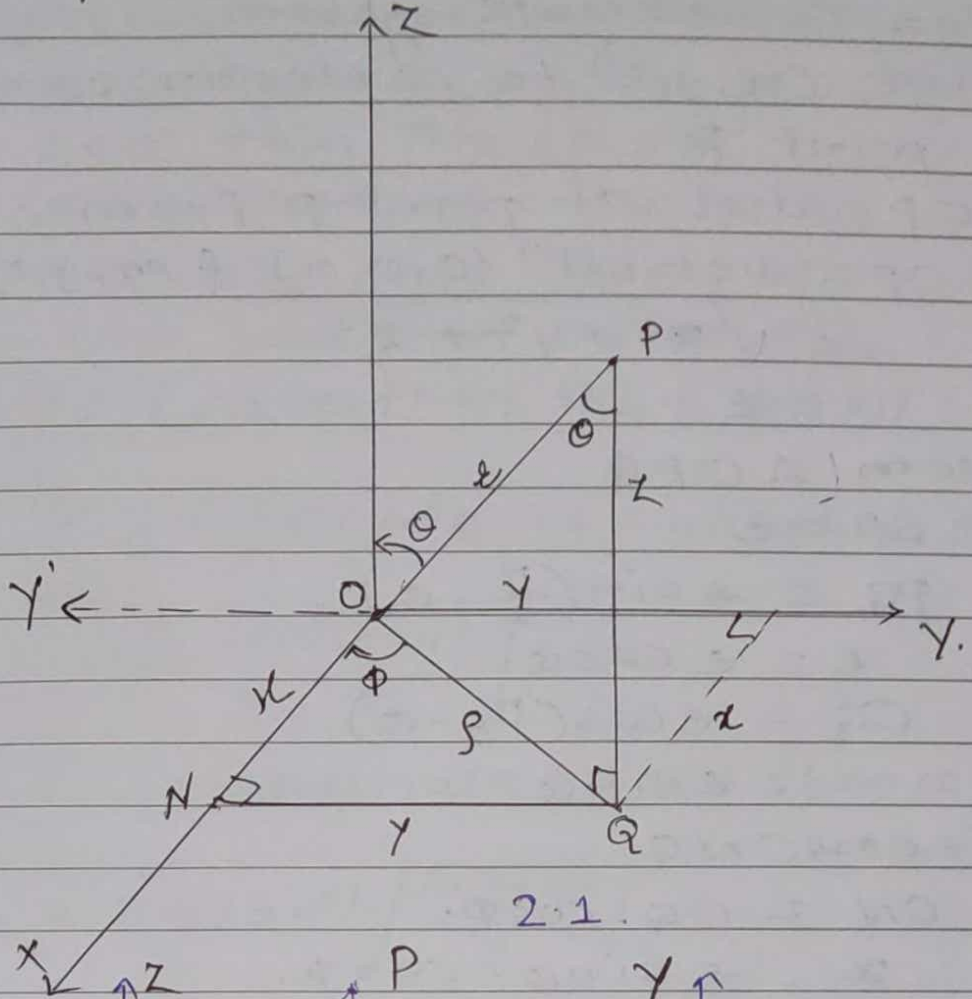
* Equation of XY plane is $Z = 0$
* Eqn of YZ plane is $X = 0$
* Eqn of ZX plane is $Y = 0$

The above three planes are called co-ordinate planes.
These divides the entire space in 8 equal parts called as octants.

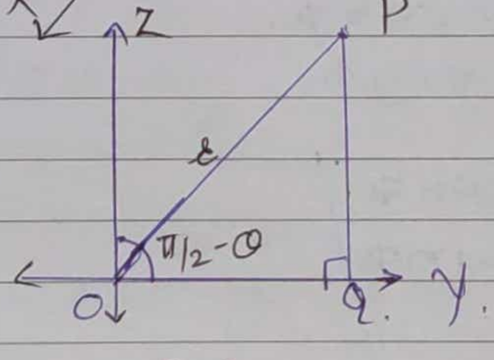
* A point P in three dimensions is denoted by three real numbers (x, y, z)



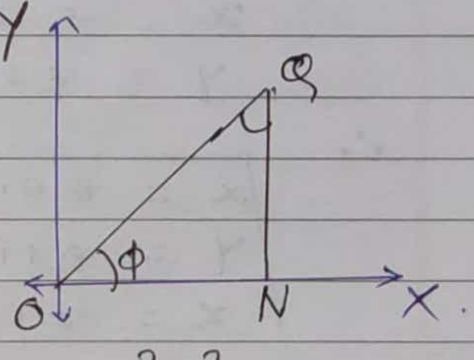
2) Spherical polar co-ordinate system:-



2.1



2.2



2.3

From fig. 2.1 let $OP = r$.

Let θ = angle made by OP with +ve z-axis

ϕ = angle made by OQ with +ve x-axis

in this system $0 < r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$.

The numbers denoted by (r, θ, ϕ) which can be associated the point P are called as sperical polar co-ordinates of pt. P.

* Relation bet^n cartesian system & spherical polar co-ordinate system.

Let (x, y, z) be cartesian co-ordinate of point P.

$\therefore$ OP = dist. of point P from origin

i.e OP = dist bet^n $(0,0,0)$ & (x, y, z)

$$= \sqrt{x^2 + y^2 + z^2}$$

$$= r$$

From ΔOPQ

$$OP = r$$

$$PQ = r\sin\left(\frac{\pi}{2} - \theta\right)$$

$$\therefore \quad z = r\cos\theta.$$

$$OQ = r\cos(\pi/2 - \theta)$$

$$= r\sin\theta.$$

& From ΔONQ

$$ON = OQ \cdot \cos\phi.$$

$$x = r\sin\theta \cdot \cos\phi.$$

$$y = r\sin\theta \cdot \sin\phi.$$

$\therefore$

$$\boxed{\begin{aligned} x &= r\sin\theta\cos\phi. \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta. \end{aligned}}$$

& also

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\& \ \theta = \tan^{-1}\left|\frac{\sqrt{x^2+y^2}}{z}\right|$$

$$\& \ \phi = \tan^{-1}\left|\frac{y}{x}\right|$$

Note :-

1] Foe spheeical polae co-oedinates.

(i) If $z > 0$ then $0 \le \theta \le \pi/2$

(ii) If $z < 0$ then $\pi/2 \le \theta \le \pi$

(iii) If $x > 0, y > 0$ then then $0 \le \phi \le \pi/2$

(iv) If $x < 0, y > 0$ then $\pi/2 \le \phi \le \pi$

(v) If $x < 0, y < 0$ then $\pi \le \phi \le 3\pi/2$

(vi) If $x > 0, y < 0$ then $3\pi/2 \le \phi \le 2\pi$.

2] If z-cooedinate is +ve then θ is acute angle

$$\theta = \tan^{-1}\left|\frac{\sqrt{x^2+y^2}}{z}\right|$$

If z-co-oedinate is -ve then θ is obtuse angle

$$\theta = \pi - \tan^{-1}\left|\frac{\sqrt{x^2+y^2}}{z}\right|.$$

3] f Foe ϕ.

2nd

$(-,+)$

$\phi = \pi - \tan^{-1}\left|\frac{y}{x}\right|$

1st quadeant.

$(+,+)$

$\phi = \tan^{-1}\left|\frac{y}{x}\right|$

3ed

$(-,-)$

$\phi = \pi + \tan^{-1}\left|\frac{y}{x}\right|$

4th.

$(+,-)$

$\phi = 2\pi - \tan^{-1}\left|\frac{y}{x}\right|$.

3] Cylindeical Polae Co-oedinates :

From Fig 2.1 let $OQ = \rho$

$\therefore PQ = z$.

Feom ΔOQN.

$ON = OQ \cdot \cos\phi$
$NQ = OQ \sin\phi$.
$\therefore x = \rho\cos\phi$.
$y = \rho\sin\phi$.
$z = z$
where $\rho = \sqrt{x^2+y^2}$
$\phi = \tan^{-1}\left|\frac{y}{x}\right|$
$\therefore$ coordinates are (ρ, ϕ, z).

Example.
① Find the spherical polar & cylindrical co-ordinates of $(-3,-4,-5) = (x,y,z)$
$\Rightarrow$ we know that
Spherical co-ordinates are, (r, θ, ϕ)
as we know, $r = \sqrt{x^2+y^2+z^2}$
$= \sqrt{(-3)^2+(-4)^2+(-5)^2}$
$= \sqrt{9+16+25}$.
$= 5\sqrt{2}$.

As z is $-ve$.
$\therefore \theta = \pi - \tan^{-1}\left|\frac{\sqrt{x^2+y^2}}{z}\right|$.

$= \pi - \tan^{-1}\left|\frac{\sqrt{9+16}}{5}\right|$

$= \pi - \tan^{-1}\left|\frac{\sqrt{25}}{5}\right|$

$= \pi - \tan^{-1}\left|\frac{5}{5}\right|$

$= \pi - \tan^{-1}|1|$

$= \pi - \frac{\pi}{4}$ $\qquad \because \tan^{-1}(1) = \pi/4$.

$= \frac{3\pi}{4}$.

$\boxed{\theta = 135^\circ}$

As, x, y both are $-ve$

$\therefore \phi = \pi + \tan^{-1}\left|\frac{y}{x}\right|$

$= \pi + \tan^{-1}\left|\frac{4}{3}\right|$

$\phi = 233°8'$

$\therefore (r, \theta, \phi) = (5\sqrt{2}, 135°, 233°8')$.

Now to find cylindrical coordinates

$\rho = \sqrt{x^2+y^2}$

$= \sqrt{9+25}$

$= 5$

$\phi = \pi + \tan^{-1}\left|\frac{y}{x}\right|$

$= 233°8'$

& $z = -5$

$\therefore (\rho, \phi, z) = (5, 233°8', -5)$.

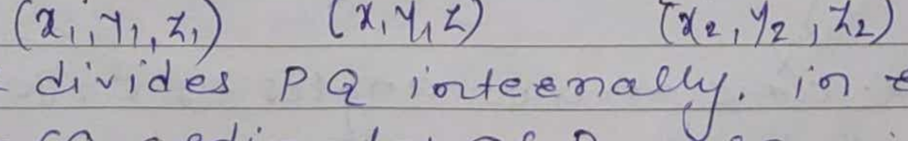
* Distance formula:

$P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$ be two points then dist. betn P & Q

is $PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

* Section Formula:

i) Internal Division:



if R divides PQ internally in ratio $m:n$.
then co-ordinates of R are given by

$x = \frac{mx_2+nx_1}{m+n}, \quad y = \frac{my_2+ny_1}{m+n}, \quad z = \frac{mz_2+nz_1}{m+n}$.

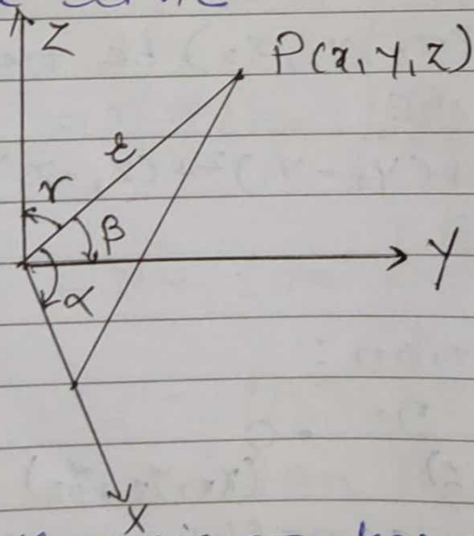
* Exteenal Division:-
If R divides PQ exteenally in the eatio. m:n then the co-oedinate of R are

$$x=\frac{mx_2-nx_1}{m-n},\ y=\frac{my_2-ny_1}{m-n},\ z=\frac{mz_2-nz_1}{m-n}$$

* Mid point Foemula:
If R is mid point of PQ then it divides PQ. in eatio 1:1
∴ co-oedinates of R are

$$x=\frac{x_2+x_1}{2},\ y=\frac{y_2+y_1}{2},\ z=\frac{z_2+z_1}{2}$$

* Dieection cosines of a line:-(DC's).
If α, β, γ are the angles made by the given line with +ve x, y & z axes eespectively then $\cos\alpha, \cos\beta, \cos\gamma$ are known as dc's of a line



As X-axis makes angles of 0, 90°, 90° with X, Y & Z axes resp.
∴ cos 0, cos 90, cos 90 are dc's of X-axis.
∴ ~~(0, 0~~ (1, 0, 0) are dc's of X-axis

Similarly
(0, 1, 0) are dc's of Y axis
(0, 0, 1) are dc's of Z-axis

* If l, m, n are dc's of a given line OP & (x, y, z) are co-ordinates of point P where OP = r then,

$$x = r\cos\alpha = lr$$
$$y = r\cos\beta = mr$$
$$z = r\cos\gamma = nr.$$

$\therefore\ l^2 + m^2 + n^2 = 1$ i.e
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$

* Angle :- betⁿ two lines.

If l_1, m_1, n_1 & l_2, m_2 & n_2 are dc's of two lines then

a) Angle betⁿ them.

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

b) Two lines are ||el if

$$l_1/l_2 = m_1/m_2 = n_1/n_2 = \text{const}.$$

c) Two lines are ⊥ar if :

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0.$$

* Direction Ratio's of a Line :-

The numbers a, b, c which are proportional to dc's l, m & n are known as directⁿ ratio's or dr's of a line.

i.e $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c} = \pm\dfrac{1}{\sqrt{a^2+b^2+c^2}}$

- i) we can calculate dc's from dr's a, b, c by using
$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}},$$
$$n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

ii) Let $A(x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$. The dr's of line AB are
$$x_2 - x_1, \; y_2 - y_1, \; z_2 - z_1$$

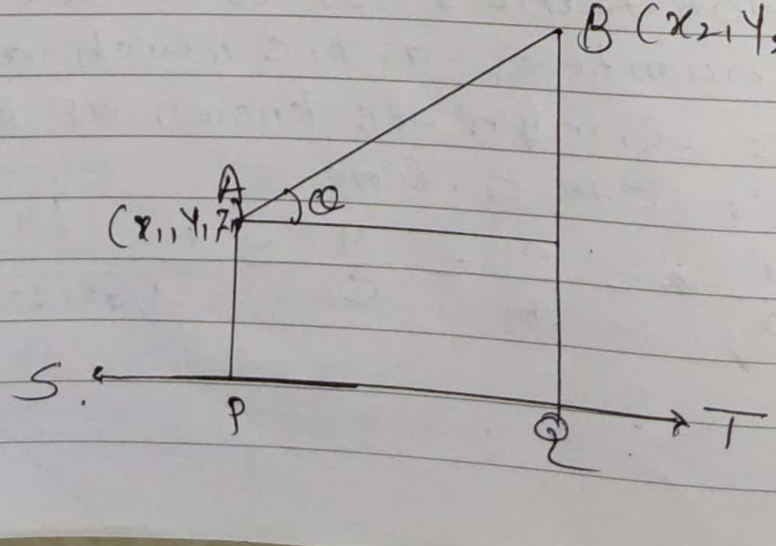
iii) If a_1, b_1, c_1 & a_2, b_2, c_2 are dr's of two lines then

a) Angle betn them :
$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

b) Two lines are llel if
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \text{const.}$$

c) Two lines are $\perp^{ar}$ if.
$$a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

* Projection formula of line segment.



Let l, m, n be dc's of line ~~AB~~ ST
Let AB be the given line segment
its projectⁿ on line ST is given by PQ

$$PQ = l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1).$$

* Equation of Plane :-

a] General Form :

$$ax + by + cz + d = 0$$

where const. a, b, c are dr's of normal to plane.

b] Passing through origin :

$$ax + by + cz = 0$$

c] Eqⁿ of plane passing through (x_1, y_1, z_1) & having a, b, c are dr's of normal to plane then eqⁿ is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

d] Intercept Form : The plane which makes intercepts a, b, c on co-ordinate axes is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

e] Normal Form : If l, m, n are dc's of normal to the plane & 'p' is the length of $\perp^{ar}$ from origin to the plane, then its eqⁿ is

$$lx + my + nz = P.$$

f] The eqⁿ of plane llel to the plane $ax+by+cz+d=0$ is

$$ax + by + cz + d_1 = 0.$$

Note: i) From any eqⁿ of plane the coeff. of x, y, z gives dr's of normal to the plane.

ii) Two planes are llel if their normals are llel.

iii) Two planes are $\perp^{ar}$ if their normals are $\perp^{ar}$.

iv) Angle betn two planes:-

$a_1x+b_1y+c_1z+d_1=0$ &

$a_2x+b_2y+c_2z+d_2=0$

θ is the angle betn ~~them~~ their normals having a_1, b_1, c_1 & a_2, b_2, c_2 respectively.

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

G] Length of perpendicular.

a] From point (x_1, y_1, z_1) to plane $ax+by+cz+d=0$ is

$$p = \left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$$

$\therefore$ From $(0,0,0)$

$$p = \left|\frac{d}{\sqrt{a^2+b^2+c^2}}\right|.$$

b] Eqn of plane passing through the intersection of two planes:-

$a_1x+b_1y+c_1z+d_1=0$ &

$a_2x+b_2y+c_2z+d_2=0$ is

$$(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0$$

where λ is parameter.

G] Eqⁿ of plane passing through Three points (x_1, y_1, z_1), (x_2, y_2, z_2) & (x_3, y_3, z_3) is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

H] Equations of a line :-

i) As st line is the intersection of two planes

ii) Two point formula:

The line joining (x_1, y_1, z_1) & (x_2, y_2, z_2) is given by

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

iii) Symmetrical Form: The line having dr's (a, b, c) & passing through (x_1, y_1, z_1) is given by

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

I] Coplanarity of two lines:

Two lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ &

$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are.

are coplanar if.

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Examples:

① Find the eqⁿ of plane which passes through the point $(3,-3,1)$ & is
(i) ||el to the plane $2x+3y+5z+6=0$
(ii) normal to the line joining the points $(3,2,-1)$ & $(2,-1,5)$.
(iii) Perpendicular to the planes
$7x+y+2z=6$ &
$3x+5y-6z=8$.

→ i) Any plane ||el to the plane is $ax+by+cz+k=0$ which goes through $(3,-3,1)$, k is any number.
∴ if we take $k=-2$
∴ ||el plane is
$ax+by+cz-2=0$.

ii) Any plane through $(3,-3,1)$ is
$a(x-3)+b(y+3)+c(z-1)=0$ — ①
[∵ eqⁿ of plane through pt (x_1,y_1,z_1) is $a(x-x_1)+(y-y_1)+(z-z_1)=0$]

∴ The dc's of line joining $(3,2,-1)$ & $(2,-1,5)$ are
$(3-2),(2-(-1)),(-1-5)$
i.e $1,3,-6$.
∴ Values of a, b, c are $1,3,-6$ resp.
∴ eqⁿ of plane is from ①

$1(x-3)+3(y+3)-6(z-1)=0$
OR
$x+3y-6z+12=0$.

iii) Any plane through $(3,-3,1)$.

$a(x-3)+b(y+3)+c(z-1)=0$ which will be $\perp^{ar}$ to the planes

$7x+y+2z=6$ &

$3x+5y-6z=8$.

Now we want to find a, b, c For that.

Let $7a+b+2c=0$

& $3a+5b-6c=0$

Solve this eq^n simultaneously we get.

$$\frac{a}{\begin{vmatrix} 1 & 2 \\ 5 & -6 \end{vmatrix}} = \frac{b}{\begin{vmatrix} 7 & 2 \\ 3 & -6 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 7 & 1 \\ 3 & 5 \end{vmatrix}}$$

$$\Rightarrow \frac{a}{(-6-10)} = \frac{-b}{(-42-6)} = \frac{c}{35-3}$$

$$\Rightarrow \frac{a}{-16} = \frac{-b}{-48} = \frac{c}{32}$$

$$\Rightarrow \frac{a}{-1} = \frac{-b}{-3} = \frac{c}{2}$$

$$\Rightarrow \frac{a}{1} = \frac{b}{-3} = \frac{c}{-2}$$

Hence by intercept Form.
eq^n of plane is,

$1(x-3)-3(y+3)-2(z-1)=0$.

OR

$x-3y-2z-10=0$.

(2) The plane $4x+5y-z=7$ is rotated through a right angle about its line of intersection with the plane $2x+3y-3z=5$. find the eq^n of this plane in its new position.

→ Any plane through the line of intersectⁿ of

$$4x+5y-z=7 \quad -①$$

& $$2x+3y-3z=5 \quad -②$$

is $4x+5y-z-7+k(2x+3y-3z-5)=0$

i.e $(4+2k)x+(5+3k)y-(1+3k)z-(7+5k)=0$ ③

Then new position of ① when rotated through right angle, is s.t ① & ③ are $\perp^{ar}$

This requires that

$$4(4+2k)+5(5+3k)-(1+3k)=0$$

$$26k+42=0 \quad \text{or} \quad k=-\frac{21}{13}$$

substituting $k=-21/13$ in ③

we get

$$10x+2y+50z+14=0$$

OR $$5x+y+25z+7=0$$

which is required plane.

Example on line:-

Q.1 Find ~~the~~ in symmetrical form, the eqⁿ of line

$$x+y+z+1=0$$
$$4x+y-2z+2=0$$

⟹ How to Reduce general eqⁿ of a line of the symmetrical form:

① Find a pt on the line by putting $z=0$ in the given eqⁿ & solving the resulting eqⁿ of x & y.

② Find the dc's of line from the fact that is $\perp^{ar}$ to the normals to the given planes.

③ Write the eqⁿ of the ~~plane~~ line in symm. form.

∴ For above example:

① To find a point on the line.

putting $z=0$ in the given eqⁿ, we have

$$x+y+1=0 \;;\; 4x+y+2=0.$$

solving, $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{1}{-3}$

∴ A point on the lines is $(-1/3, -2/3, 0)$.

② To find dc's l, m, n of line.

∵ lines lies on both the planes.

∴ It is $\perp^{ar}$ to their normals whose dr's are proportional to $(1,1,1)$ & $(4,1,-2)$

i.e $l+m+n=0 \;;\; 4l+m-2n=0$

solving, ∴ $\dfrac{l}{\begin{vmatrix}1 & 1\\ 1 & -2\end{vmatrix}} = \dfrac{-m}{\begin{vmatrix}1 & 1\\ 4 & -2\end{vmatrix}} = \dfrac{n}{\begin{vmatrix}1 & 1\\ 4 & 1\end{vmatrix}}$

∴ $\dfrac{l}{-2-1} = \dfrac{-m}{-2-4} = \dfrac{n}{1-4}$

∴ $\dfrac{l}{-3} = \dfrac{m}{6} = \dfrac{n}{-3}$

$\Rightarrow \dfrac{l}{-1} = \dfrac{m}{2} = \dfrac{n}{-1}$

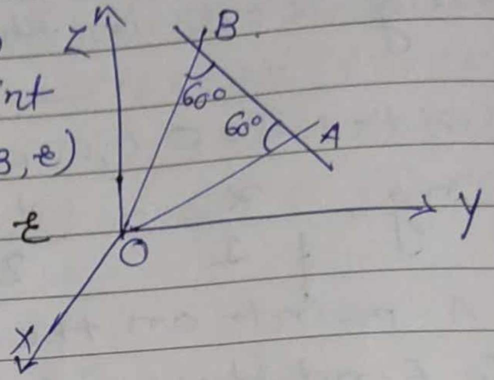
∴ dc's are $-1, 2, -1$.

Thus the eqⁿ of line in the symmetrical form: $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c}$

$$\frac{x+1/3}{-1} = \frac{y+2/3}{2} = \frac{z}{-1}$$

Q.2 Find the eqn of the two st. line through the origin, each of which intersect the st. line $\frac{1}{2}(x-3) = (y-3) = z$ & is inclined at an angle of $60°$ to it.

$\Rightarrow$ Let AB be the given line so that any point A on it is $(2t+3, t+3, t)$



$\left[\because \frac{(x-3)}{2} = \frac{y-3}{1} = \frac{z}{1} = t\right.$

$\therefore x = 2t+3$

$y = t+3$

$\left. z = t \right]$

$\therefore$ Dr's of OA are $(2t+3-0, t+3-0, t-0)$

Angle betn AO & AB has to be $60°$

$$\therefore \cos 60° = \frac{2(2t+3) + 1(t+3) + 1(t)}{\sqrt{2^2+1^2+1^2}\sqrt{(2t+3)^2+(t+3)^2+t^2}}$$

OR

$$\frac{1}{2} = \frac{6t+9}{\sqrt{6(6t^2+18t+18)}}$$

squaring

$$\frac{1}{4} = \frac{(6t+9)^2}{6(6t^2+18t+18)}$$

$\Rightarrow 3\cancel{6}(6t^2+18t+18) = \overset{2}{\cancel{4}}(6t+9)^2$

$\Rightarrow 18t^2+54t+54 = \cancel{12t}\, 2(36t^2+108t+81)$

$18t^2+54t+54 = 72t^2+2\times108t+2\times81$

$9t^2+27t+27 = 36t^2+108t+81$

$t^2+3t+3 = 4t^2+12t+9$

$4t^2-t^2+12t-3t+9-3=0$

$3t^2+9t+6=0$

$\Rightarrow r^2 + 3r + 2 = 0$

i.e $r = -1, -2$

$\therefore$ co-ordinates of A & B are $(1, 2, -1)$ & $(-1, 1, -2)$

Hence eqn of the required lines OA & OB. are $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ & $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$.

* Eqn of plane through the line

eqn of line $\Rightarrow \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$.

eqn of plane is.

$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$

where, $al + bm + cn = 0$.

Q.3 Find the eqn in the symmetrical form of the projection of the line $\frac{x-1}{2} = -(y+1) = \frac{z-3}{4}$ on the plane $x+2y+z=12$

$\Rightarrow$ Any plane through the given line is,

$A(x-1) + B(y+1) + C(z-3) = 1$ — ①

$2A - B + 4C = 0$. — ②

The plane ① will be $\perp^{ar}$ to the given plane, if $A + 2B + C = 0$ — ③

Solving ② & ③.

we get, $\frac{A}{-9} = \frac{B}{2} = \frac{C}{5}$

put in ① we get $9x - 2y - 5z + 4 = 0$ — ④.

which cuts the given plane $x+2y+z=12$ — ⑤ along the required line of projection one point on the line is got by putting $z=0$ in ④ & ⑤

& solving

$\Rightarrow \quad 9x-2y-5(0)+4=0$

$x+2y+0=12$

$\Rightarrow \quad 9x-2y+4=0$

$x+2y=12$

$\Rightarrow$ point is $\left(\frac{4}{5}, \frac{28}{5}, 0\right)$.

The dr's of the line are found by solving

$l+2m+n=0$ & $9l-2m-5n=0$

to be $4, -7, 10$.

Hence the required eqn of line of projection are,

$$\frac{x-4/5}{4} = \frac{y-28/5}{-7} = \frac{z}{10}.$$

Q.4 Show that the lines $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ &

$$\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$$ are coplanar

find their common point & the eqn of the plane in which they lie.

$\Rightarrow$ Any point on the first line is,

$(5+4t, 7+4t, -3-5t)$ —①

which lies on the second line if.

$$\frac{-3+4t}{7} = \frac{3+4t}{1} = \frac{-8-5t}{3} \quad —②$$

$\Rightarrow$ from ②

$$\frac{-3+4t}{7} = 3+4t \Rightarrow \boxed{t=-1}$$

This value satisfies $\frac{3+4t}{7} = \frac{-8-5t}{3}$.

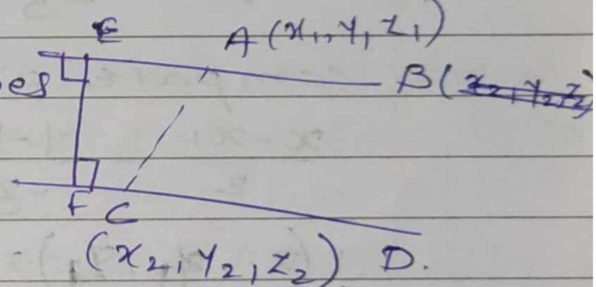
$\Rightarrow$ lines intersect (i.e coplanar) & from ① point of intersectⁿ is $(1, 3, 2)$.
The eqⁿ of plane in which they lie is

$$\begin{vmatrix} x-5 & y-7 & z+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0$$

i.e $17x - 47y - 24z + 172 = 0$.

* Shortest distance between Two lines :-

Two st lines which do not lie in one plane are called skew lines. Such lines possess a common $\perp^{ar}$ which is shortest dist. betⁿ them.



Let the given skew lines AB & CD be

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$$

& $$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$$

Let l, m, n be the dc's of shortest dist. EF.

$\because$ EF $\perp$ to both AB & CD.

$\therefore$ $ll_1 + mm_1 + nn_1 = 0$ & $ll_2 + mm_2 + nn_2 = 0$

solving

& To find eqⁿ of line of shortest dist. we observe that it is coplanar with both AB & CD.

plane containing the lines AB & EF is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \quad \text{— (A)}$$

plane containing the lies CD & EF is

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0. \quad \text{— (B)}$$

(A),(B) are eqn of line of shortest dist.

Q. Find the magnitude & the eqn of the shortest dist betn the lines

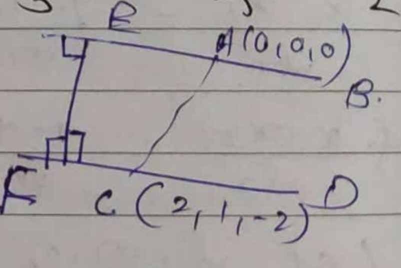
$$\frac{x}{2} = \frac{y}{-3} = \frac{z}{1} \ \& \ \frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$$

⇒ compare with

$$\frac{x-x_1}{2} = \frac{y-y_1}{-3} = \frac{z-z_1}{1} \ \& \ \frac{x-x_2}{3} = \frac{y-y_2}{-5} = \frac{z-z_2}{2}$$

$\therefore (x_1, y_1, z_1) \equiv (0,0,0)$ &

$(x_2, y_2, z_2) \equiv (2,1,-2)$.



Let l, m, n be dc's of shortest distance EF.

$\because$ EF $\perp$ to both AB & CD.

$\therefore 2l - 3m + n = 0 \quad , \quad 3l - 5m + 2n = 0$

Solving

$$\therefore \frac{l}{\begin{vmatrix} -3 & 1 \\ -5 & 2 \end{vmatrix}} = \frac{-m}{\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 2 & -3 \\ 3 & -5 \end{vmatrix}}$$

$$\therefore \frac{l}{(-6+5)} = \frac{-m}{(4-3)} = \frac{n}{(-10+9)}$$

$$\therefore \frac{l}{-1} = \frac{-m}{1} = \frac{n}{-1}$$

$$\frac{l}{1} = \frac{m}{1} = \frac{n}{1} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}$$

$\therefore$ Length of S.D(EF) = projection of AC on EF.

$$= (2-0)\frac{1}{\sqrt{3}} + (1-0)\frac{1}{\sqrt{3}} + (-2-0)\cdot\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

The eq^n of the line of shortest dist (EF) are

$$\begin{vmatrix} x & y & z \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 \;\&\; \begin{vmatrix} x-2 & y-1 & z+2 \\ 3 & -5 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

i.e $4x+y-5z=0$ & $7x+y-8z=31$.

* Intersect^n of Three planes :-

Any three planes (no two of which are ||el) intersect in one of the following ways:

① The planes may meet in a point, if the line of section of two of them is not ||el to third

② The planes may have common line of sec^n if the line of sect^n of two of them lies on the third.

③ The planes may form triangular prism, if the line of sect^n of two of them is ||el to third but does not lie on it.

Sphere

Defn :- A sphere is locus of a point which remains at a const dist from fixed point.

The fixed point is called centre & the const dist is called radius of sphere.

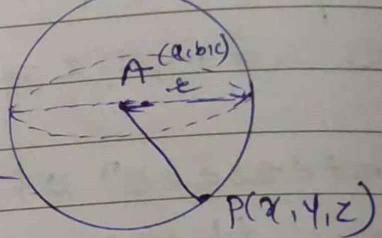
* Equations of sphere:

1) Centre radius form:

Let $P(x, y, z)$ be any point on the sphere

let $AP = r$.

$\therefore (AP)^2 = r^2$



$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

2) Standard form:

centre is $(0,0,0)$ then the eqn of the sphere is

$x^2 + y^2 + z^2 = r^2$

3) General form:

The general form of eqn of sphere is

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

4) Diameter form:

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$

& AB = diameter of sphere.

$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0$

5) Intercept form:

$x^2 + y^2 + z^2 - ax - by - cz = 0$

Example.

① Obtain the eqn of sphere passing through the four points.

$(0,-2,4)$, $(3,1,4)$, $(1,2,3)$ & $(4,-4,2)$

$\Rightarrow$ Let the eqn of the sphere be,

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ①

it passes through (0, -2, 4)

$\Rightarrow 20 - 4v + 8w + d = 0$ — (2)

it passes through (3, 1, 4)

$\Rightarrow 26 + 6u + 2v + 8w + d = 0$ — (3)

it passes through (1, 2, 3)

$\Rightarrow 14 + 2u + 4v + 6w + d = 0$ — (4)

it passes through (4, -4, 2)

$\Rightarrow 36 + 8u - 8v + 4w + d = 0$ — (5)

solving eqn (2), (3), (4) & (5) simultaneously.

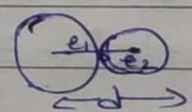
we get $u = -2,\ v = 1,\ w = -1,\ d = -8$

$\therefore$ required eqn of sphere is

$$x^2 + y^2 + z^2 - 4x + 2y - 2z - 8 = 0.$$

* Touching Spheres (Externally & internally).

(1) Two spheres are said to be touch externally if the distance betn their ceters is equal to sum of their radii



i.e $d = r_1 + r_2$

(2) & for internally $d = |r_1 - r_2|$

Example:-

(1) Find the eqn of sphere passing through (1, 0, 0), (0, 1, 0), (0, 0, 1) & having least possible radius.

$\Rightarrow$ Let $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ be eqn of sphere.

This passes through (1, 0, 0)

$\Rightarrow 1 + 2u + d = 0$

i.e $u = \dfrac{(d+1)}{-2}$

Similarly $v = \frac{1+d}{-2}$, $w = \frac{1+d}{-2}$
&

we know

$$r^2 = u^2+v^2+w^2-d$$

$$\therefore r^2 = \frac{3(d+1)^2}{4} - d$$

$$= \frac{3d^2+2d+3}{4}$$

$$r^2 = f(d)$$

& we know that for least possible radius $f'(d)=0$

i.e $6d+2=0$

$$\boxed{d = -\frac{1}{3}}$$

$$\therefore 2u = 2v = 2w = -\frac{2}{3}.$$

$\therefore$ eq^n is

$$x^2+y^2+z^2-\frac{2}{3}(x+y+z)-\frac{1}{3}=0.$$

(2) A sphere of radius r passes through the origin & meets the axes in A, B, C show that the locus of centroid of triangle ABC is
$9(x^2+y^2+z^2)=4r^2$

$\Rightarrow$ Let
$A(x_1,0,0)$, $B(0,y_1,0)$, $C(0,0,z_1)$ be the co-ordinates. Then eq^n of sphere OABC in intercept form is

$$x^2+y^2+z^2-xx_1-yy_1-zz_1=0 \quad \text{—(1)}$$

its centre is

$$\left(\frac{x_1}{2}, \frac{y_1}{2}, \frac{z_1}{2}\right)$$

& radius $r = \sqrt{\frac{x_1^2}{4}+\frac{y_1^2}{4}+\frac{z_1^2}{4}}$

$$r^2 = \frac{x_1^2+y_1^2+z_1^2}{4}$$

$$\Rightarrow 4r^2 = x_1^2+y_1^2+z_1^2 \quad \text{—— (2)}.$$

Let $(\bar{x},\bar{y},\bar{z})$ be point on the locus
i.e $\bar{x},\bar{y},\bar{z}$ be the centroid of ABC.

$\therefore \bar{x} = \frac{x_1+0+0}{3}, \ \bar{y} = \frac{0+y_1+0}{3}, \ \bar{z} = \frac{0+0+z_1}{3}$

($\because$ centroid point $= (x = \frac{x_1+x_2+x_3}{3}, y = \frac{y_1+y_2+y_3}{3}, z = z_1+z_2+z_3/3)$)

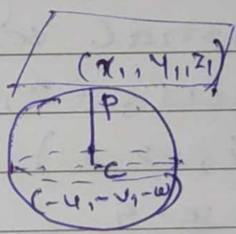
$\therefore x_1 = 3\bar{x}, \ y_1 = 3\bar{y}$ & $z_1 = 3\bar{z}$

put in eqn (2).

$$\therefore 9(\bar{x}^2+\bar{y}^2+\bar{z}^2) = 4r^2$$

Now just replace $\bar{x},\bar{y},\bar{z}$ by x,y,z
we get

$$\boxed{9(x^2+y^2+z^2) = 4r^2}$$



* Tangent Plane :-

• The eqn of tangent plane to the sphere:

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$$

at (x_1,y_1,z_1) is given by.

$$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$$

Notes:-
To find eqn of tangent plane to the sphere at (x_1,y_1,z_1)

Replace x^2 by xx_1
y^2 by yy_1
z^2 by zz_1

Replace $2x$ by $x+x_1$
$2y$ by $y+y_1$
$2z$ by $z+z_1$ in the eqn of sphere

Q.1 Find the eqn of tangent plane & normal to the sphere $x^2+y^2+z^2-4x+2y-4=0$ at $(4,-2,2)$

⇒ Given sphere is

$$x^2+y^2+z^2-4x+2y-4=0$$

its tangent plane at (x_1, y_1, z_1) is

$$xx_1+yy_1+zz_1-2(x+x_1)+(y+y_1)-4=0$$

Given, $(x_1, y_1, z_1)=(4,-2,2)$

∴ $4x+(-2)y+2z-2(x+4)+(y-2)-4=0$

i.e. $2x-y+2z-14=0$.

is the eqn of tangent plane.

Now to find Normal,

we know that coeff of x, y, z from eqn of tangent plane gives dr's of normal to the plane

Thus, $(2,-1,2)$ are dr's of normal & $(4,-2,2)$ is one pt on it

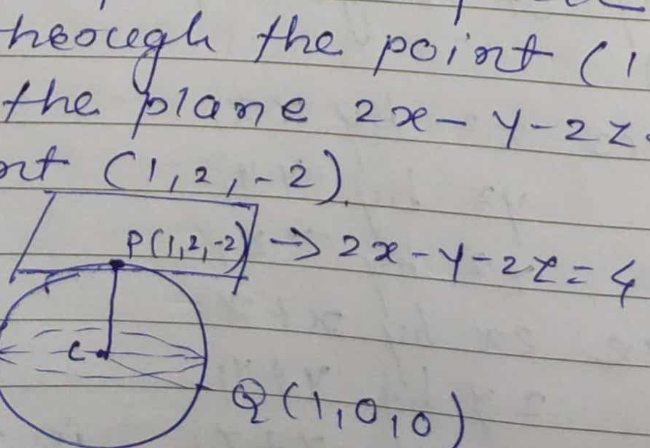
∴ $$\frac{x-4}{2}=\frac{y+2}{-1}=\frac{z-2}{2}$$

is the eqn of normal line.

* Sphere Touching the Given plane:-

Q① Find the eqn of the sphere which passes through the point $(1,0,0)$ & touches the plane $2x-y-2z=4$ at the point $(1,2,-2)$.

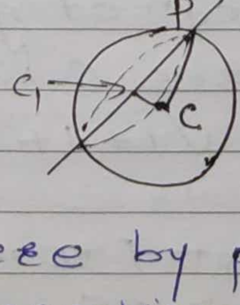
→



⇒ Given plane is
$2x - y - 2z = 4$.
∴ coeffs. of (x, y, z) i.e $(2, -1, 2)$ are
dr's of the normal CP
& co-ordinates of P are $(1, 2, -2)$.
Thus eqⁿ of CP is
$$\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z+2}{2} = t \text{ (say)}$$
∴ $x = 2t+1$, $y = 2-t$, $z = -2t-2$
but from fig. $(CP)^2 = (CQ)^2$
$$\therefore (2t+1-1)^2 + (-t+2-2)^2 + (-2t-2+2)^2$$
$$= (2t+1-1)^2 + (-t+2)^2 + (-2t-2)^2$$
$$\Rightarrow 4t^2 + t^2 + 4t^2 = 4t^2 + t^2 - 4t + 4 + 4t^2 + 8t$$

$$\Rightarrow \boxed{t = -2}$$
∴ co-ordinates of 'C' are $(-3, 4, 2)$
& $(CQ)^2 = (-3-1)^2 + (4-0)^2 + (2-0)^2$
$= 36$
∴ we can use centre radius form.
$$(x+3)^2 + (y-4)^2 + (z-2)^2 = 36$$
$$\therefore x^2 + y^2 + z^2 + 6x - 8y - 4z - 7 = 0.$$

* The section of a sphere by a plane.



Note:-
1] The sectⁿ of sphere by plane gives a circle.
2] The curve of intersection of two spheres is also a circle.
3] The sectⁿ of sphere by a plane through the centre of the sphere is called great circle

Its centre & radius are same as that of the given sphere.

4] If $S_1=0$ & $S_2=0$ are two spheres then $S_1-S_2=0$ is plane in which circle lies.

Called radical plane.

5] If $S=0$, $U=0$ are sphere & plane then together represents circle.

6] If $S_1=0$ & $S_2=0$ are spheres then together represents circle.

7] Sphere through circle:

If $S=0$ & $U=0$ represent circle then $S+\lambda U=0$ is family of spheres passing through circle.

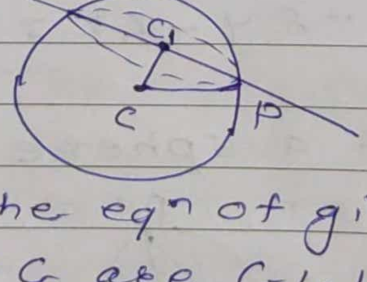
Example:- centre & radius of circle :-

Q. Find the centre & radius of the circle.

$x^2+y^2+z^2+2x-2y-4z-19=0$

$x+2y+2z+7=0$

$\Rightarrow$



From the eq^n of given sphere co-ordinates of C are $(-1,1,2)$ &

radius $= CP = \sqrt{1+1+4+19} = 15$

& $CC_1 = \perp^{r}$ dist from C $(-1,1,2)$ to plane

$x+2y+2z+7=0$

$$\therefore CC_1 = \left|\frac{-1+2+4+7}{\sqrt{1+4+4}}\right| = 4$$

Now in ΔCPC_1

$$CP^2 = CC_1^2 + C_1P^2$$

$$\therefore C_1P^2 = CP^2 - CC_1^2$$

$$= 25 - 16$$

$$= 9$$

$\therefore$ Radius of circle $= 3$

To find coordinates of C_1, find eqn of CC_1

From eqn of plane

$$x + 2y + 2z + 7 = 0$$

Coeff of x, y, z are dr's of normal to ~~the~~.

$\therefore (1, 2, 2)$ are dr's of CC_1 & $C(-1, 1, 2)$ is one of point CC_1

thus, $\dfrac{x+1}{1} = \dfrac{y-1}{2} = \dfrac{z-2}{2} = t$

$x = t - 1, \quad y = 2t + 1, \quad z = 2t + 2$

is any pt on CC_1

$\therefore$ it lies on plane also

$\therefore$ put in eqn of plane.

$\therefore (t-1) + 2(2t+1) + 2(2t+2) + 7 = 0$

$\Rightarrow t = -4/3$

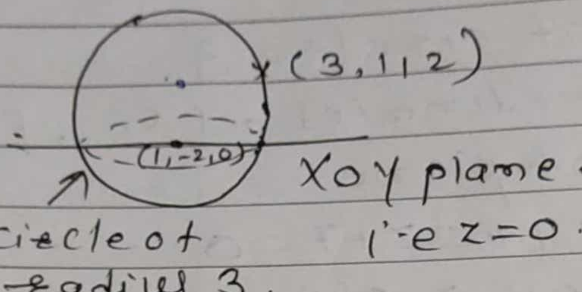
$\therefore C_1 = \left(-7/3, -5/3, -2/3\right)$ = centre of circle

Example: Sphere passing through the circle:-

1) $S + \lambda U = 0$ represent sphere which passes through circle $S = 0$, $U = 0$

2) $S_1 + \lambda S_2 = 0$ represents the sphere which passes through the circle $S_1 = 0$, $S_2 = 0$.

Example: Find the eqn of sphere which passes through $(3,1,2)$ & meets XOY plane in a circle of radius 3 units with the centre at $(1,-2,0)$



From Fig. eqn of circle in XOY plane whose centre is $(1,-2,0)$ & radius 3 is

$(x-1)^2+(y+2)^2=9$

Thus eqn of sphere whose intersectn with $z=0$ of above circle is

$(x-1)^2+(y+2)^2+z^2=9.$

$\therefore x^2+y^2+z^2-2x+4y-4=0$

Thus the required sphere is

$S+\lambda U=0$

$x^2+y^2+z^2-2x+4y-4+\lambda z=0$ — (1)

which passes through $(3,1,2)$

$\therefore$ satisfies eqn (1).

$\therefore 9+1+4-6+4-4+2\lambda=0$

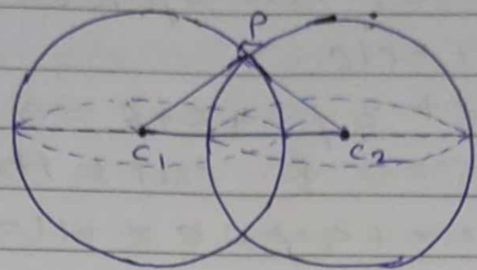
$\Rightarrow \boxed{\lambda=-4}$

$\therefore$ eqn (1) becomes

$x^2+y^2+z^2-2x+4y-4z-4=0$

required eqn of sphere.

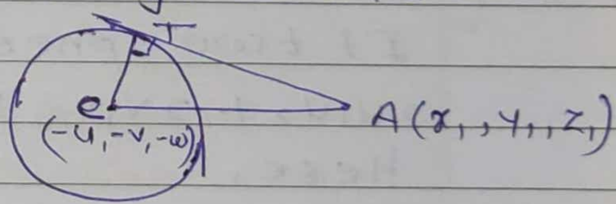
Orthogonal Spheres :-



Two spheres are said to be orthogonal if the tangent planes to the two spheres at the points of intersection are at right angles.
i.e Normals to planes are $\perp^{ar}$.
cond^n of orthogonality:
$$2(u_1u_2 + v_1v_2 + w_1w_2) = d_1 + d_2.$$

* Length of the tangent line:-

Let $A(x_1, y_1, z_1)$ be any point outside the sphere. $x^2+y^2+z^2+2ux+2vy+2wz+d=0$
Then length of tangent line from A to the sphere is given by



$$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d.$$
it can solved by, ΔATO

$$CT = \text{radius} = \sqrt{u^2+v^2+w^2-d}$$
& $AT^2 = \cancel{CA^2}\ CA^2 - CT^2$

* Radical Plane:-

if S_1 & S_2 are two spheres then eq^n of Radical plane is
$$S_1 - S_2 = 0.$$

Example

1. Find the eq^n of the sphere that passes through the circle.
$x^2+y^2+z^2-2x+3y-4z+6=0$ &
$3x-4y+5z-15=0$ & cuts the sphere
$x^2+y^2+z^2+2x+4y-6z+11=0$
Orthogonally.

$\Rightarrow$ Eq^n of sphere passing circle.

$$(x^2+y^2+z^2-2x+3y-4z+6)+\lambda(3x-4y+5z-15)=0 \quad (1)$$

$$\Rightarrow x^2+y^2+z^2+x(3\lambda-2)+y(3-4\lambda)+z(5\lambda-4)+6-15\lambda=0$$

The above sphere is orthogonal to the sphere
i.e. $x^2+y^2+z^2+2x+4y-6z+11=0$

If two spheres are orthogonal then,

$$2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2 \quad (2)$$

Here,

$2u_1=3\lambda-2$, $2v_1=3-4\lambda$
$2w_1=5\lambda-4$, $d_1=6-15\lambda$
$u_2=1$, $v_2=2$, $w_2=-3$ & $d_2=11$.

put in eq^n (2) we get.

$$3\lambda-2+6-8\lambda-15\lambda+12=17-15\lambda$$

$$\Rightarrow -5\lambda=1$$

$$\boxed{\lambda=-1/5}$$

Substitute in (1)

$\therefore$ eq^n of sphere is

$$5(x^2+y^2+z^2)-13x+19y-25z+45=0$$